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THE INVARIANTS, SEMINVARIANTS AND LINEAR COVARIANTS OF THE BINARY QUARTIC FORM MODULO 2.

By L. E. Dickson.

Denote the quartic form by

$$(1)\qquad ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4,$$

where $a, \cdots, e$ are undetermined integers taken modulo 2 so that $a^2 \equiv a$, etc. Upon replacing x by $x + y$ in (1), we obtain a new quartic with the following coefficients modulo 2:

$$(2)\qquad a' = a, \quad b' = b, \quad c' = b + c, \quad d' = b + d, \quad e' = a + b + c + d + e.$$

A polynomial $P(a, \cdots, e)$ with integral coefficients is called a seminvariant of (1) modulo 2 if

$$P(a', \cdots, e') \equiv P(a, \cdots, e) \qquad (\text{mod } 2).$$

Theorem 1. *Every seminvariant is a rational integral function of*

$$(3)\qquad a, \quad b, \quad \alpha = (b - 1)c, \quad \beta = c + d, \quad \gamma = ac + cd + be, \quad \delta = (b - 1)(a + c + d - 1)e.$$

The functions (3) are easily verified by means to (2) to be seminvariants. The theorem will follow* if we show that these six seminvariants serve to characterize the classes of quartics (1) under the set of two transformations

$$(T)\qquad x \equiv x' + ty', \qquad y \equiv y' \qquad (\text{mod } 2).$$

Two quartics are said to be in the same class if and only if they are transformable into each other by transformations of type (T).

If $b \equiv 1$, we apply (T) for $t = c$ to (1) and get

$$(4)\qquad ax^4 + x^3y + \beta xy^3 + \gamma y^4.$$

If $b \equiv 0$, $a + c + d \equiv 1$, we apply (T) for $t = e$ to (1) and get

$$(5)\qquad ax^4 + \alpha x^2y^2 + (\alpha + \beta)xy^3 \qquad (a + \beta \equiv 1).$$

If $b \equiv 0$, $a + c + d \equiv 0$, (1) becomes

$$(6)\qquad ax^4 + \alpha x^2y^2 + (\alpha + \beta)xy^3 + \delta y^4 \qquad (a + \beta \equiv 0).$$

Since the forms (4)–(6) involve only the seminvariants, the theorem follows.

Theorem 2. *A complete set of linearly independent seminvariants is*

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* American Journal of Mathematics, vol. 31 (1909), p. 337.

given by the following twenty:

$$(7)\qquad \delta,\ \delta a,\ \delta\alpha,\ \delta a\alpha,\qquad b,\ ba,\ b\beta,\ ba\beta,$$

$$(8)\qquad 1,\ a,\ \alpha,\ a\alpha,\qquad \beta,\ \beta a,\ \beta\alpha,\ \beta a\alpha,\qquad \gamma,\ \gamma a,\ \gamma\beta,\ \gamma a\beta.$$

The number of forms (4)–(6) is $8 + 4 + 8$, so that there are 20 classes. Hence there are exactly 20 linearly independent seminvariants.* But

$$(9)\ \delta b \equiv 0,\quad \delta\beta \equiv \delta a,\quad \delta\gamma \equiv \delta\alpha,\quad b\alpha \equiv 0,\quad (b-1)\gamma \equiv \alpha\gamma \equiv \alpha(1 + a + \beta).$$

Hence any polynomial in the functions (3) is a linear function of the eight functions (7) and those combinations (8) of a, α, β, γ which lack $\alpha\gamma$. As a check, we may verify that the twenty functions (7), (8) are linearly independent modulo 2.

A seminvariant unaltered by the substitution $(ae)(bd)$, induced by the interchange of x and y, is an invariant. Obvious invariants† are

$$(10)\qquad L = b + c + d = b + \beta,\qquad Q = c^2 + bd = \alpha + b\beta,$$
$$M = ace(b-1)(d-1) = a\alpha\delta.$$

From the product of the first two we obtain

$$(11)\ J = (b-1)(c-1)(d-1) = (L+1)(Q+1) = (\beta+1)(b+\alpha+1).$$

General theorems‡ give this J and

$$(12)\qquad I = (a-1)(e-1)J = (a-1)(J + \alpha\delta + \delta).$$

There are two further invariants of the second order:

$$(13)\qquad V = ad + be + cL = \alpha + \gamma + a\beta,$$
$$W = ae + (a+e)(L+1) = \delta + a + ab + a\gamma + \beta\gamma + a\beta.$$

From these we get§

$$(14)\qquad WL = aeL = a(\delta + \gamma + \beta\gamma),$$
$$VL + V = abd + acd + bce + bde = a\alpha + \beta\gamma + ab\beta.$$

The preceding nine invariants have been expressed linearly in terms of the seminvariants (7), (8). From the general linear function of the latter we subtract constant multiples of

$$(15)\qquad I,\ M,\ WL,\ W,\ VL,\ J,\ V,\ Q,\ L,\ 1$$

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[*] Trans. Amer. Math. Soc., vol. 10 (1909), p. 126.

[†] The algebraic invariants $4I$ and $16J$ of (1) reduce modulo 2 to Q and $V + LQ$. See also end of paper.

[‡] Trans. Amer. Math. Soc., vol. 8 (1907), pp. 206–7, § 1, § 3.

[§] The eliminant of (1), $x^2 \equiv x$, $y^2 \equiv y$ is aeL. See preceding paper, § 4.

to eliminate in turn the terms in

$$\alpha\delta,\ a\alpha\delta,\ a\delta,\ \delta,\ \beta\gamma,\ \alpha\beta,\ \gamma,\ b\beta,\ \beta,\ 1.$$

It therefore remains to test for invariance a function

$$A = k_1b + k_2ba + k_3ba\beta + k_4a + k_5\alpha + k_6a\alpha + k_7\beta a + k_8\beta a\alpha + k_9\gamma a + k_{10}\gamma a\beta, \tag{16}$$

in which the k's are constants. Applying the substitution $(ae)(bd)$, we get

$$\begin{aligned} A' = k_1d + k_2de + k_3de(c + b) + k_4e + k_5(d - 1)c + k_6(d - 1)ce \\ + k_7(c + b)e + k_8(b + 1)(d - 1)ce + k_9\gamma'e + k_{10}\gamma'e(c + b), \\ \gamma' \equiv ce + bc + ad. \end{aligned} \tag{17}$$

By the terms free of e, $k_i = 0$ $(i = 1, \cdots, 8)$, $k_9 = k_{10}$. Then $k_9 \equiv 0$.

THEOREM 3. *The ten functions* (15) *form a complete set of linearly independent invariants. They are linearly independent of the ten seminvariants in* (16).

The invariants (15) are functions of L, Q, I, V, W since

$$M = I + J + W(Q + L + 1), \qquad J = (L + 1)(Q + 1). \tag{18}$$

Any polynomial in these five can be expressed as a linear function of the invariants (15) by use of the relations (18) and

$$WV = VL + V + WL, \qquad QV = VL + V + QL, \qquad x^2 \equiv x,\ Ix \equiv 0, \tag{19}$$

where x is any one of the five. By the linear independence of the functions (15) we thus obtain

THEOREM 4. *As a fundamental system of invariants we may take L, Q, I, V, W. No one of these is congruent to a rational integral function of the remaining four.*

A linear covariant of (1) must evidently be of the form

$$C = (i + A)x + (i + A')y,$$

where i is an invariant, while A and A' are given by (16) and (17). This C has the covariant property with respect to the interchange of x and y. Its covariance with respect to $x \equiv x' + y'$, $y \equiv y'$, requires that

$$(i + A)(x + y) + (i + E)y \equiv C,$$

where E is the function derived from A' by the substitution (2). Thus

$$i = A + A' + E.$$

Since A is a seminvariant, this i is evidently unaltered by (2). It will therefore be an invariant if unaltered by the substitution $(ae)(bd)$. The latter condition is found to give

$$i = k_2(V + L) + k_3(VL + V) + k_4L + k_5Q + k_6(LQ + VL + V) + k_7V + (k_8 + k_9 + k_{10})LQ.$$

Denote by K_i the covariant obtained by setting $k_i \equiv 1$, $k_j \equiv 0 (k \neq i)$. These covariants are linear combinations of products of

$$\lambda = bx + dy, \qquad \mu = (L + a)x + (L + e)y \tag{20}$$

(viz., K_1, K_4) by invariants. Indeed,

$$K_5 = (Q + \alpha)x + (Q + cd + c)y = b(c + d)x + d(c + b)y = (L + 1)\lambda = Q\lambda,$$

$$K_7 = (V + \beta a)x + (V + ce + be)y = W\lambda + V\mu,$$

$$K_2 + K_7 + \mu = (L + 1)(ax + ey) = (L + 1)\mu = W\mu, \qquad K_3 = (L + 1)K_7,$$

$$K_6 + K_3 = Q\mu = V\lambda + K_7,$$

$$K_8 = c(b + 1)(d + 1)\{(a + 1)x + (e + 1)y\} = QW\mu + K_6 = (VL + V)\lambda + K_3 + K_6,$$

$$K_9 = V(\lambda + \mu) + K_3, \qquad K_{10} = K_9 + WL\lambda.$$

THEOREM 5. *The ten linearly independent linear covariants are functions of the two covariants* (20) *and the invariants.*

In addition to the syzygies between (20) and invariants which are included in the preceding set of relations, we note

$$I\lambda = M\lambda = J\lambda = I\mu = M\mu = 0.$$

The resultants of λ, μ; λ, quartic (1); μ, (1) are respectively $V + L$ $V + LQ$, $L + WL$. The modular invariants of λ and μ are respectively $(b - 1)(d - 1) \equiv Q+L + 1$, $W + L + 1$.

UNIVERSITY OF CHICAGO,
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